DE GRUYTER
OPEN

DOI 10.1515/tmj-2017-0042

Certain classes of bi-univalent functions with bounded boundary variation

H. Orhan[1], N. Magesh[2] and V. K. Balaji[3]

[1]Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey
[2]Post-Graduate and Research Department of Mathematics, Government Arts College for Men, Krishnagiri 635001, Tamilnadu, India
[3]Department of Mathematics, L.N. Govt College, Ponneri, Chennai, Tamilnadu, India
E-mail: orhanhalit607@gmail.com[1], nmagi_2000@yahoo.co.in[2], balajilsp@yahoo.co.in[3]

Dedicated to Prof. Dr. Muhammet Kamali on the occasion of his fifty sixth anniversary.

Abstract

In their pioneering work dated 2010 on the subject of bi-univalent functions, Srivastava et al. actually revived the study of the coefficient problems involving bi-univalent functions in recent years. Inspired by the pioneering work of Srivastava et al., there has been triggering interest to study the coefficient problems for many different subclasses of bi-univalent functions. Motivated largely by a number of sequels to the investigation by Srivastava et al., in this paper, we consider certain classes of bi-univalent functions to obtain the estimates of their second and third Taylor-Maclaurin coefficients. Further, certain special cases are also indicated. Some interesting remarks about the results presented here are also discussed.

2010 Mathematics Subject Classification. **30C45**. 30C50
Keywords. Univalent functions, bi-univalent functions, bi-starlike function, bi-convex function, functions with bounded boundary rotation, coefficient estimates, convolution (Hadamard) product.

1 Introduction and definitions

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe one-quarter theorem [9] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains the disc with the center in the origin and the radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$, satisfying $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$ and

$$f\left(f^{-1}(w)\right) = w,\ |w| < r_0(f),\ r_0(f) \geq \frac{1}{4}.$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots,\ w \in f(\mathbb{D}). \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent, if both f and f^{-1} are univalent in $\mathbb{D}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{D}$, and we denote by Σ this class of bi-univalent functions.

Tbilisi Mathematical Journal 10(4) (2017), pp. 17–27.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 31 January 2017.
Accepted for publication: 25 August 2017.

Recently, in their pioneering work on the subject of bi-univalent functions, Srivastava et al. [28] actually revived the study of the coefficient problems involving bi-univalent functions. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35] and references therein). The aforecited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [28]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \dots)$ for functions $f \in \Sigma$ is still an open problem.

Definition 1.1. [18] Let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions P, normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} P(z) - \beta}{1 - \beta} \right| \mathrm{d}\,\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{D}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions p analytic in $\mathbb{D}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}}\, \mathrm{d}\,\mu(t), \tag{1.3}$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m,\ m \geq 2. \tag{1.4}$$

Clearly, $\mathcal{P} := \mathcal{P}_2$ is the well-known class of Carathéodory functions. That is, the normalized functions with positive real part in the open unit disc $\mathbb{D}$.

Definition 1.2. A function $f \in \Sigma$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belongs to the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\alpha \geq 0$, $\lambda \geq 0$, $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left((1 - \alpha + 2\lambda)\frac{f(z)}{z} + (\alpha - 2\lambda)f'(z) + \lambda z f''(z) - 1 \right) \in \mathcal{P}_m(\beta),\ z \in \mathbb{D} \tag{1.5}$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma}\left((1 - \alpha + 2\lambda)\frac{g(w)}{w} + (\alpha - 2\lambda)g'(w) + \lambda w g''(w) - 1 \right) \in \mathcal{P}_m(\beta),\ w \in \mathbb{D}. \tag{1.6}$$

It is interesting to note that the special values of α, γ, λ, β and m lead the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$ to various subclasses, we illustrate the following subclasses:

1. For $\alpha = 1 + 2\lambda$, we get the class $\mathcal{W}_\Sigma(\gamma, \lambda, 1 + 2\lambda, m; \beta) \equiv \mathcal{F}_\Sigma(\gamma, \lambda, m; \beta)$. A function $f \in \Sigma$ of the form
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
is said to be in $\mathcal{F}_\Sigma(\gamma, \lambda, m; \beta)$, if the following conditions
$$1 + \frac{1}{\gamma}\left(f'(z) + \lambda z f''(z) - 1\right) \in \mathcal{P}_m(\beta),\ z \in \mathbb{D}$$
and for $g(w) = f^{-1}(w)$
$$1 + \frac{1}{\gamma}\left(g'(w) + \lambda w g''(w) - 1\right) \in \mathcal{P}_m(\beta),\ w \in \mathbb{D}$$
hold.

2. For $\lambda = 0$, we obtain the class $\mathcal{W}_\Sigma(\gamma, 0, \alpha, m; \beta) \equiv \mathcal{B}_\Sigma(\gamma, \alpha, m; \beta)$. A function $f \in \Sigma$ of the form
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
is said to be in $\mathcal{B}_\Sigma(\gamma, \alpha, m; \beta)$, if the following conditions
$$1 + \frac{1}{\gamma}\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right) \in \mathcal{P}_m(\beta),\ z \in \mathbb{D}$$
and for $g(w) = f^{-1}(w)$
$$1 + \frac{1}{\gamma}\left((1-\alpha)\frac{g(w)}{w} + \alpha g'(w) - 1\right) \in \mathcal{P}_m(\beta),\ w \in \mathbb{D}$$
hold.

   Remark 1.3. For $\gamma = 1$ and $m = 2$ the class $\mathcal{B}_\Sigma(1, \alpha, 2; \beta) \equiv \mathcal{B}_\Sigma(\alpha; \beta)$ was introduced and studied by Frasin and Aouf [10].

3. For $\lambda = 0$ and $\alpha = 1$, we have the class $\mathcal{W}_\Sigma(\gamma, 0, 1, m; \beta) \equiv \mathcal{P}_\Sigma(\gamma, m; \beta)$. A function $f \in \Sigma$ of the form
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
is said to be in $\mathcal{P}_\Sigma(\gamma, m; \beta)$, if the following conditions
$$1 + \frac{1}{\gamma}\left(f'(z) - 1\right) \in \mathcal{P}_m(\beta),\ z \in \mathbb{D}$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma}\left(g'(w) - 1\right) \in \mathcal{P}_m(\beta),\ w \in \mathbb{D}$$

hold.

Remark 1.4. For $\gamma = 1$ and $m = 2$, the class $\mathcal{P}_\Sigma(1, 2; \beta) \equiv \mathcal{P}_\Sigma(\beta)$ was introduced and studied by Srivastava et al. [28] (see [12]).

Definition 1.5. A function $f \in \Sigma$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belongs to the class $\mathcal{R}_\Sigma(\gamma, \lambda, m; \beta)$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda \geq 0$, $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} - 1\right) \in \mathcal{P}_m(\beta),\ z \in \mathbb{D} \tag{1.7}$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma}\left(\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} - 1\right) \in \mathcal{P}_m(\beta),\ w \in \mathbb{D}. \tag{1.8}$$

Remark 1.6. For $\gamma = 1$ and $m = 2$, the class $\mathcal{R}_\Sigma(1, \lambda, 2; \beta) \equiv \mathcal{R}_\Sigma(\lambda, \beta)$ was introduced and studied by Prema and Keerthi [19].

1. For $\lambda = 0$, we have the class $\mathcal{R}_\Sigma(\gamma, 0, m; \beta) \equiv \mathcal{S}_\Sigma(\gamma, m; \beta)$. A function $f \in \Sigma$ of the form

   $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

   is said to be in $\mathcal{S}_\Sigma(\gamma, m; \beta)$, if the following conditions

   $$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \in \mathcal{P}_m(\beta),\ z \in \mathbb{D}$$

   and for $g(w) = f^{-1}(w)$

   $$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)} - 1\right) \in \mathcal{P}_m(\beta),\ w \in \mathbb{D}$$

   hold.

In order to prove our results for the function in the classes $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$ and $\mathcal{R}_\Sigma(\gamma, \lambda, m; \beta)$, we need the following lemma due to Goswami et al. [11]:

Lemma 1.7. Let the function $\Phi(z) = 1 + \sum\limits_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{D}$, such that $\Phi \in \mathcal{P}_m(\beta)$. Then,

$$|h_n| \leq m(1 - \beta),\ n \geq 1.$$

In this investigation, we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the subclass $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$ and $\mathcal{R}_\Sigma(\gamma, \lambda, m; \beta)$. Also, we obtain the upper bounds using the results of $|a_2|$ and $|a_3|$.

2 Main results

In the following theorem, we obtain coefficient estimates for functions in the class $f \in \mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$.

Theorem 2.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$. Then

$$|a_2| \leq \min \left\{ \sqrt{\frac{|\gamma|\, m(1-\beta)}{1+2\alpha+2\lambda}};\ \frac{|\gamma|\, m(1-\beta)}{1+\alpha} \right\}, \tag{2.1}$$

$$|a_3| \leq \frac{|\gamma|\, m(1-\beta)}{1+2\alpha+2\lambda}, \tag{2.2}$$

$$\left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\, m(1-\beta)}{1+2\alpha+2\lambda}. \tag{2.3}$$

Proof. Since $f \in \mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m; \beta)$, from the Definition 1.2 we have

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z} + (\alpha - 2\lambda) f'(z) + \lambda z f''(z) - 1\right) = p(z) \tag{2.4}$$

and

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w} + (\alpha - 2\lambda) g'(w) + \lambda w g''(w) - 1\right) = q(w), \tag{2.5}$$

where $p,\ q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Using the fact that the functions p and q have the following Taylor expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots,\ z \in \mathbb{D}, \tag{2.6}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots,\ w \in \mathbb{D}, \tag{2.7}$$

and equating the coefficients in (2.4) and (2.5), from (1.2) we get

$$\frac{1}{\gamma}(1+\alpha) a_2 = p_1, \tag{2.8}$$

$$\frac{a_3}{\gamma}(1+2\alpha+2\lambda) = p_2, \tag{2.9}$$

$$-\frac{1}{\gamma}(1+\alpha) a_2 = q_1, \tag{2.10}$$

and

$$\frac{(1+2\alpha+2\lambda)}{\gamma}(2a_2^2 - a_3) = q_2. \tag{2.11}$$

Since $p,\ q \in \mathcal{P}_m(\beta)$, according to Lemma 1.7, the next inequalities hold:

$$|p_k| \leq m(1-\beta),\ k \geq 1, \tag{2.12}$$

$$|q_k| \leq m(1-\beta),\ k \geq 1, \tag{2.13}$$

and thus, from (2.9) and (2.11), by using the inequalities (2.12) and (2.13), we obtain

$$|a_2|^2 \leq |\gamma| \frac{|q_2| + |p_2|}{2\left[1 + 2\alpha + 2\lambda\right]} \leq \frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha + 2\lambda},$$

which gives

$$|a_2| \leq \sqrt{\frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha + 2\lambda}}. \tag{2.14}$$

From (2.8), by using (2.12) we obtain immediately that

$$|a_2| = \left| \frac{\gamma p_1}{1 + \alpha} \right| \leq \frac{|\gamma|\, m(1-\beta)}{1 + \alpha},$$

and combining this with the inequality (2.14), the first inequality of the conclusion is proved.

According to (2.9), from (2.12) we easily obtain

$$|a_3| = \left| \frac{\gamma p_2}{1 + 2\alpha + 2\lambda} \right| \leq \frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha + 2\lambda},$$

and from (2.11), by using (2.13) we finally deduce

$$\left|2a_2^2 - a_3\right| = \left| \frac{\gamma q_2}{1 + 2\alpha + 2\lambda} \right| \leq \frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha + 2\lambda},$$

which completes our proof. ∎

Corollary 2.2. Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{F}_\Sigma(\gamma, \lambda, m; \beta)$. Then

$$|a_2| \leq \min\left\{ \sqrt{\frac{|\gamma|\, m(1-\beta)}{3 + 6\lambda}};\ \frac{|\gamma|\, m(1-\beta)}{2 + 2\lambda} \right\},$$

$$|a_3| \leq \frac{|\gamma|\, m(1-\beta)}{3 + 6\lambda}, \quad \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\, m(1-\beta)}{3 + 6\lambda}.$$

Corollary 2.3. Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{B}_\Sigma(\gamma, \alpha, m; \beta)$. Then

$$|a_2| \leq \min\left\{ \sqrt{\frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha}};\ \frac{|\gamma|\, m(1-\beta)}{1 + \alpha} \right\},$$

$$|a_3| \leq \frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha}, \quad \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\, m(1-\beta)}{1 + 2\alpha}.$$

Corollary 2.4. Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{P}_\Sigma(\gamma, m; \beta)$. Then

$$|a_2| \leq \min\left\{ \sqrt{\frac{|\gamma|\, m(1-\beta)}{3}};\ \frac{|\gamma|\, m(1-\beta)}{3} \right\},$$

$$|a_3| \leq \frac{|\gamma|\, m(1-\beta)}{3}, \quad \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\, m(1-\beta)}{3}.$$

Setting $\beta = 0$ in Theorem 2.1 we get the following special case:

Corollary 2.5. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, m)$. Then

$$|a_2| \leq \min\left\{\sqrt{\frac{|\gamma|\,m}{1+2\alpha+2\lambda}};\ \frac{|\gamma|\,m}{1+\alpha}\right\},\ |a_3| \leq \frac{|\gamma|\,m}{1+2\alpha+2\lambda},\ \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\,m}{1+2\alpha+2\lambda}.$$

For $\beta = 0$ the Corollary 2.2 reduces to the next result:

Example 2.6. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{F}_\Sigma(\gamma, \lambda, m; 0)$ then

$$|a_2| \leq \min\left\{\sqrt{\frac{|\gamma|\,m}{3+6\lambda}};\ \frac{|\gamma|\,m}{2+2\lambda}\right\},\ |a_3| \leq \frac{|\gamma|\,m}{3+6\lambda},\ \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\,m}{3+6\lambda}.$$

For $\beta = 0$ the Corollary 2.3 reduces to the next result:

Example 2.7. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{B}_\Sigma(\gamma, \alpha, m; 0)$ then

$$|a_2| \leq \min\left\{\sqrt{\frac{|\gamma|\,m}{1+2\alpha}};\ \frac{|\gamma|\,m}{1+\alpha}\right\},\ |a_3| \leq \frac{|\gamma|\,m}{1+2\alpha},\ \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\,m}{1+2\alpha}.$$

For $\beta = 0$ the Corollary 2.4 reduces to the next result:

Example 2.8. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{P}_\Sigma(\gamma, m; 0)$, then

$$|a_2| \leq \sqrt{\frac{m}{3}},\quad |a_3| \leq \frac{m}{3},\quad \text{and}\quad \left|2a_2^2 - a_3\right| \leq \frac{m}{3}.$$

If we put $\gamma = 1$ and $m = 2$ in Corollary 2.4, we have the following corollary:

Corollary 2.9. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{B}(\beta)$, then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & \text{if}\quad 0 \leq \beta \leq \frac{1}{3}, \\ 1-\beta, & \text{if}\quad \frac{1}{3} < \beta < 1, \end{cases}\qquad |a_3| \leq \frac{2(1-\beta)}{3},\quad \text{and}\quad \left|2a_2^2 - a_3\right| \leq \frac{2(1-\beta)}{3}.$$

Remark 2.10. For the special case $\frac{1}{3} < \beta < 1$, the above first inequality, and the second one for all $0 \leq \beta < 1$, improve the estimates given by Srivastava et al. in [28, Theorem 2].

Theorem 2.11. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{R}_{\Sigma}(\gamma, \lambda, m; \beta)$. Then

$$|a_2| \leq \min\left\{\sqrt{\frac{|\gamma|\, m(1-\beta)}{2+\lambda}};\ \frac{|\gamma|\, m(1-\beta)}{1+\lambda}\right\}, \tag{2.15}$$

$$|a_3| \leq \frac{|\gamma|\, m(1-\beta)}{2+\lambda}, \tag{2.16}$$

$$\left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\, m(1-\beta)}{2+\lambda}. \tag{2.17}$$

Proof. Since $f \in \mathcal{R}_{\Sigma}(\gamma, \lambda, m; \beta)$, from the Definition 1.5 we have

$$1 + \frac{1}{\gamma}\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} - 1\right) = p(z) \tag{2.18}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} - 1\right) = q(w), \tag{2.19}$$

where $p,\ q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Using the fact that the functions p and q have the following Taylor expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots,\ z \in \mathbb{D}, \tag{2.20}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots,\ w \in \mathbb{D}, \tag{2.21}$$

and equating the coefficients in (2.18) and (2.19), from (1.2) we get

$$\frac{1}{\gamma}(1+\lambda)a_2 = p_1, \tag{2.22}$$

$$\frac{a_3}{\gamma}(2+\lambda) = p_2, \tag{2.23}$$

$$-\frac{1}{\gamma}(1+\lambda)a_2 = q_1, \tag{2.24}$$

and

$$\frac{(2+\lambda)}{\gamma}(2a_2^2 - a_3) = q_2. \tag{2.25}$$

Since $p,\ q \in \mathcal{P}_m(\beta)$, according to Lemma 1.7, the next inequalities hold:

$$|p_k| \leq m(1-\beta),\ k \geq 1, \tag{2.26}$$

$$|q_k| \leq m(1-\beta),\ k \geq 1, \tag{2.27}$$

and thus, from (2.23) and (2.25), by using the inequalities (2.26) and (2.27), we obtain

$$|a_2|^2 \leq |\gamma|\, \frac{|q_2| + |p_2|}{2\,(2+\lambda)} \leq \frac{|\gamma|\, m(1-\beta)}{2+\lambda},$$

which gives

$$|a_2| \leq \sqrt{\frac{|\gamma|\, m(1-\beta)}{2+\lambda}}. \tag{2.28}$$

From (2.22), by using (2.26) we obtain immediately that

$$|a_2| = \left|\frac{\gamma p_1}{1+\lambda}\right| \leq \frac{|\gamma|\, m(1-\beta)}{1+\lambda},$$

and combining this with the inequality (2.28), the first inequality of the conclusion is proved.

According to (2.23), from (2.26) we easily obtain

$$|a_3| = \left|\frac{\gamma p_2}{2+\lambda}\right| \leq \frac{|\gamma|\, m(1-\beta)}{2+\lambda},$$

and from (2.25), by using (2.27) we finally deduce

$$\left|2a_2^2 - a_3\right| = \left|\frac{\gamma q_2}{2+\lambda}\right| \leq \frac{|\gamma|\, m(1-\beta)}{2+\lambda},$$

which completes our proof. ∎

Corollary 2.12. Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_\Sigma(\gamma, \lambda, m; \beta)$. Then

$$|a_2| \leq \min\left\{\sqrt{\frac{|\gamma|\, m(1-\beta)}{2}};\ |\gamma|\, m(1-\beta)\right\},$$

$$|a_3| \leq \frac{|\gamma|\, m(1-\beta)}{2} \quad \text{and} \quad \left|2a_2^2 - a_3\right| \leq \frac{|\gamma|\, m(1-\beta)}{2}.$$

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramanian, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. **25** (2012), no. 3, 344–351.

[2] Ş. Altınkaya and S. Yalçin, Second Hankel determinant for bi-starlike functions of order β, Matematiche (Catania) **71** (2016), no. 1, 115–125.

[3] Ş. Altınkaya and S. Yalçin, Coefficient problem for certain subclasses of bi-univalent functions defined by convolution, Math. Morav. **20** (2016), no. 2, 15–21.

[4] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad J. Math. **43** (2013), no. 2, 59–65.

[5] S. Bulut, N. Magesh and V. K. Balaji, Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions, C. R. Math. Acad. Sci. Paris **353** (2015), no. 2, 113–116.

[6] M. Çağlar, E. Deniz, H. M. Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, Turkish J. Math. **41** (2017), 694-706

[7] M. Çağlar, H. Orhan and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat, **27** (2013), no. 7, 1165–1171.

[8] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Anal. **2** (2013), no. 1, 49–60.

[9] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften Series, 259, Springer Verlag, New York, 1983.

[10] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24** (2011), no. 9, 1569–1573.

[11] P. Goswami, B. S. Alkahtani and T. Bulboacă, Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions, *arXiv:1503.04644v1 [math.CV]* March (2015).

[12] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Pan Amer. Math. J.* **22** (2012), no. 4, 15–26.

[13] J. M. Jahangiri, S. G. Hamidi and S. Abd. Halim, Coefficients of bi-univalent functions with positive real part derivatives, Bull. Malays. Math. Sci. Soc. (2) **37** (2014), no. 3, 633–640.

[14] J. M. Jahangiri, N. Magesh and J. Yamini, Fekete-Szegö inequalities for classes of bi-starlike and bi-convex functions, Electron. J. Math. Anal. Appl., **3** (2015), no. 1, 133–140.

[15] N. Magesh, T. Rosy and S. Varma, Coefficient estimate problem for a new subclass of biunivalent functions, J. Complex Anal. **2013**, Art. ID 474231, 3 pp.

[16] H. Orhan, N. Magesh and V. K. Balaji, Initial coefficient bounds for a general class of bi-univalent functions, Filomat, **29** (2015), no. 6, 1259–1267.

[17] H. Orhan, N. Magesh and J. Yamini, Bounds for the second Hankel determinant of certain bi-univalent functions, Turkish J. Math. **40** (2016), no. 3, 679–687.

[18] K. S. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math., **31** (1975), 311–323.

[19] S. Prema, B. S. Keerthi, Coefficient bounds for certain subclasses of analytic functions, Journal of Mathematical Analysis, **4** (2013), no. 1, 22–27.

[20] C. Ramachandran, R. Ambrose Prabhu and N. Magesh, Initial coefficient estimates for certain subclasses of bi-univalent functions of Ma-Minda type, Applied Mathematical Sciences, **9** (2015), no. 47, 2299 – 2308.

[21] H. M. Srivastava and D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc. **23** (2015), no. 2, 242–246.

[22] H. M. Srivastava, S. Bulut, M. Çağlar, N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat, **27** (2013), no. 5, 831–842.

[23] H. M. Srivastava, S. S. Eker and R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat **29** (2015), no. 8, 1839–1845.

[24] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some subclasses of M-fold symmetric bi-univalent functions, Acta Univ. Apulensis Math. Inform. **41** (2015), 153–164.

[25] H. M. Srivastava, S. Gaboury and F. Ghanim, Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Math. Sci. Ser. B Engl. Ed. **36** (2016), no. 3, 863–871.

[26] H. M. Srivastava, S. B. Joshi, S. S. Joshi and H. Pawar, Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, Palest. J. Math. **5** (2016), Special Issue: 1, 250–258.

[27] H. M. Srivastava, N. Magesh and J. Yamini, Initial coefficient estimates for bi-$\lambda-$ convex and bi-$\mu-$ starlike functions connected with arithmetic and geometric means, Electronic J. Math. Anal. Appl. **2** (2014), no. 2, 152 – 162.

[28] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23** (2010), no. 10, 1188–1192.

[29] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, Global J. Math. Anal. **1** (2013), no. 2, 67–73.

[30] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient bounds for a sub-class of m-fold symmetric bi-univalent functions, Tbilisi Math. J. **7** (2014), no. 2, 1–10.

[31] H. Tang, G-T Deng and S-H Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, J. Inequal. Appl. **317** (2013), 1–10.

[32] H. Tang, H. M. Srivastava, S.Sivasubramanian and P. Gurusamy, The Fekete-Szegö functional problems for some subclasses of m-fold symmetric bi-univalent functions, J. Math. Inequal. **10** (2016), no. 4, 1063–1092.

[33] A. E. Tudor, Bi-univalent functions connected with arithmetic and geometric means, J. Global Res. Math. Archives, **1** (2013), no. 3, 78–83.

[34] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. **25** (2012), no. 6, 990–994.

[35] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput. **218** (2012), no. 23, 11461–11465.